

Teaching materials for an elementary differential geometry course using KeTCindy

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Abstract

Curves and surfaces in 3-dimensional Euclidean space are studied in elementary differential geometry courses, and so effective visual teaching materials are indispensable. By using KeTCindy, we can present such materials to explain the concepts of the curvatures of curves and surfaces. Materials created with KeTCindy contain animation and are given in PDF slide format, so they are available in many environments. In this paper, we introduce these materials and their use in the classroom.

Editor's Note: The animations in this paper can be displayed properly only in Adobe Acrobat or Adobe Reader.

1 Introduction

Differential geometry of curves and surfaces is a classical, important and interesting subject in mathematics, so it should be learned by every student of advanced mathematics, not only by those who are majoring in mathematics. In fact, differential geometry is used to study curved surfaces and spaces which are the foundations of Einstein's theory of relativity. Moreover, it is applied to many other fields, for example, architecture, diagnostic imaging, game software development, data mining, etc.

In order to facilitate the learning of differential geometry, the use of mathematics software is beneficial. By using computers, one can calculate geometric invariants like curvatures in sufficiently general cases and realize geometric objects with high precision graphics. In particular, since curves and surfaces in 3-dimensional Euclidean space are visualizable in principle, there should be visual teaching materials to illustrate the concepts of curvatures of curves and surfaces without using complicated mathematical formulae.

Some attempts to explain differential geometry utilizing mathematics software have already been made. For example, in the famous book [5] by Alfred Gray, Mathematica is used to calculate and visualize geometric invariants like curvatures. See [6, 7] for other examples using Mathematica and other attempts at visualizing geometrical concepts [8, 9, 10, 11]. In this paper, we propose new

visual teaching materials for differential geometry made by using KeTCindy . The advantage of using KeTCindy is that we can obtain $\text{T}_\text{E}\text{X}$ format files of generated figures, so we can naturally insert them into PDF documents, which can be viewed by any PDF viewer without additional software.

The aim of this paper is to introduce our new teaching materials and to show how we used them in the classroom. The first author has an elementary differential geometry course for students majoring in information science or physics but not mathematics, so he tries to explain geometrical concepts using not only mathematical expressions but also concrete examples and illustrations. He used to draw many figures on the blackboard in his class but found it was difficult to express how geometric objects move or change depending on the position from which they are viewed, an essential method of explaining some concepts of “differential” geometry. Another difficulty was that some students tried to make copies of the figures in their notebooks, which took too much time and they could not keep up with the lectures. The authors believe that the use of KeTCindy can solve these difficulties. In fact, KeTCindy can draw mathematically accurate figures and produce animations as well. Furthermore, because those figures are made in PDF formats, we can share them with students on websites.

This paper is organized as follows. In the next section, we review what KeTCindy is and what can be done with it. In Section 3, we introduce our teaching materials for a course in differential geometry. Section 4 is a report on the use of these materials in class.

2 KeTCindy

KeTCindy [3, 4] is a macro package of Cinderella [1], a dynamic geometry software. To generate a figure with KeTCindy , one can use the graphical user interface of Cinderella to define points and describe the geometrical object to be drawn by CindyScript, a script language implemented in Cinderella. By clicking the “Figure” button on the Cinderella screen, a batch process is executed so that an R and/or C program is made and called to generate the $\text{L}_\text{A}\text{T}_\text{E}\text{X}$ code for the figure and then the $\text{L}_\text{A}\text{T}_\text{E}\text{X}$ compiler and a PDF viewer are called. CindyScript programs can also call other software programs such as Maxima, Risa/Asir, Fricas, Meshlab, if necessary (Figure 1). See [13, 14, 15] for details.

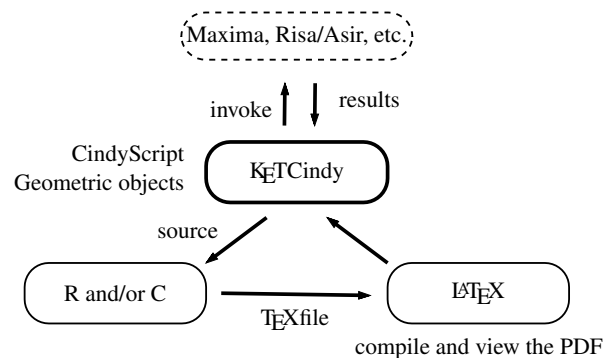


Figure 1: KeTCindy flowchart

The KeTCindy package contains various kinds of samples, so one can easily use KeTCindy simply by choosing samples and arranging them into the desired format. In the following, we review three functions of KeTCindy that we mention in the next section, i.e., animation, PDF slides, and KetCindyJS.

2.1 Animation

$\text{K}_{\text{E}}\text{T}_{\text{C}}\text{indy}$ can create PDF files with animations. In principle, $\text{K}_{\text{E}}\text{T}_{\text{C}}\text{indy}$ generates a collection of frames, figures slightly different from each other. $\text{K}_{\text{E}}\text{T}_{\text{C}}\text{indy}$ supports two types of animation, "Anime" and "Flip", both of which can be incorporated into a PDF file.

To create an Anime type animation, the collection of frames is compiled into one figure on the page and the result requires Adobe Acrobat Reader™ for playback. Below is an example of an Anime type:

Figure 2: Anime type animation

On the other hand, in a Flip type animation, each frame is sent separately (Figure 3). To play the animation, each frame is displayed separately on the page. Of course, this type of animation can be played on any kind of PDF viewer.

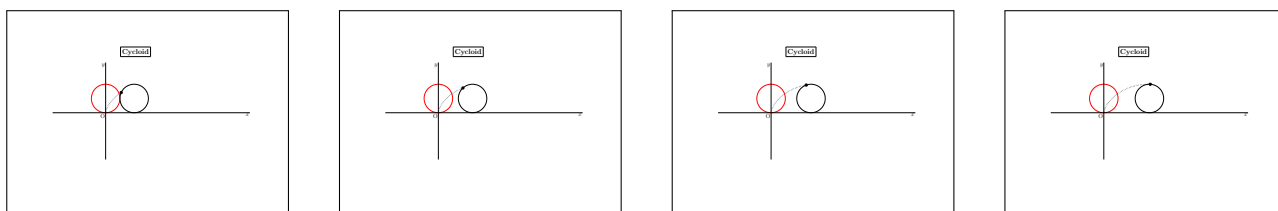


Figure 3: Flip type animation

Using Anime type animation is suitable when one wants to present the motion of a full picture, while Flip type animation may be better for presenting the motion with pauses and reverse playbacks, as one can operate the animation by oneself.

2.2 PDF slides

$\text{K}_{\text{E}}\text{T}_{\text{C}}\text{indy}$ enables users to generate PDF slides for presentations easily. To generate PDF slides, one needs a text file written in a simplified version of $\text{T}_{\text{E}}\text{X}$ format. In fact, the simplified format accepts the normal $\text{T}_{\text{E}}\text{X}$ codes, but some of the codes often used can be written in abbreviated forms. By clicking the "Slides" button, $\text{K}_{\text{E}}\text{T}_{\text{C}}\text{indy}$ generates and compiles a $\text{T}_{\text{E}}\text{X}$ source file into a PDF file. One can make lists of items to be shown one by one, like Beamer. Below is an example of a source code and the resulting slides (Figure 4):

```

1 | title::slide0::%wallpaper//
2 |
3 | %%%%%%%%%%%%%%%%%%%%%%%%%//

```

```

4 | new::Curvatures of surfaces//
5 | %repeat=4//
6 |
7 | enumerate//
8 | %thin[2,-]::item::Normal curvature//
9 | %thin[3,-]::item::Principal curvature//
10 | %thin[4,-]::item::Gaussian curvature and mean curvature//
11 | end//

```

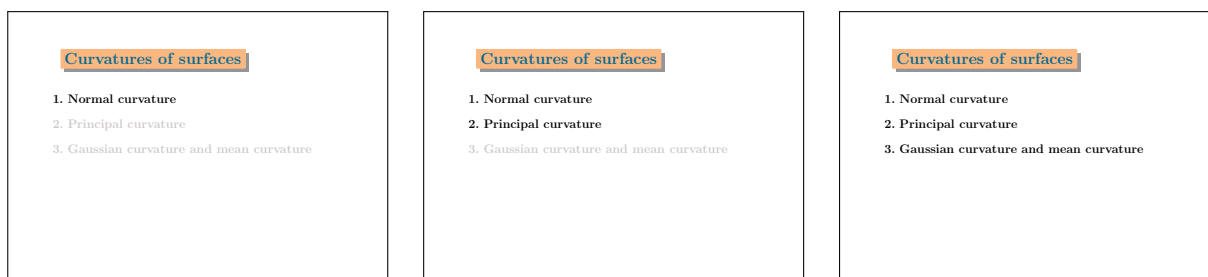


Figure 4: PDF slides

The meanings of the codes are as follows:

- $\ell.1$ inserts the title page created separately.
- $\ell.4$ displays the title of the page.
- $\ell.5$ generates a page with 4 slides. To be precise, 4 pages with the same page number will be generated.
- $\ell.7-11$ is the same as $\backslash\text{begin}\{\text{enumerate}\} \dots \backslash\text{end}\{\text{enumerate}\}$ of normal $\text{T}_{\text{E}}\text{X}$ codes.
- $\ell.8-10$ are items of the enumerate environment, but by putting $\%thin[n,-]::$ at the beginning of the line, the letters are displayed in light gray until page n , and from page n the letters are displayed in the default color, black.

The insertion into a PDF slide of animations made using $\text{K}_{\text{E}}\text{T}_{\text{C}}\text{indy}$ is very easy. In fact, to insert a Flip type animation, it is enough to add only one line like the following:

```

1 | %repeat=,para=foldername:{0}:s{25}{10}:input//

```

Then all figures in `foldername` subfolder will be displayed in the preferred position (in this case, $x\text{-position}=25$ and $y\text{-position}=10$), which will be a Flip type animation.

2.3 KeTCindyJS

CindyJS is a framework that enables CindyScript to be executed in HTML files. In particular, one can convert Cinderella content into an HTML file which can be operated on web browsers, including those of tablet devices and smartphones. Functions added as $\text{K}_{\text{E}}\text{T}_{\text{C}}\text{indy}$ can also be converted into an HTML file, which we call $\text{K}_{\text{E}}\text{T}_{\text{C}}\text{indyJS}$ (see Figure 5 and [S1]).

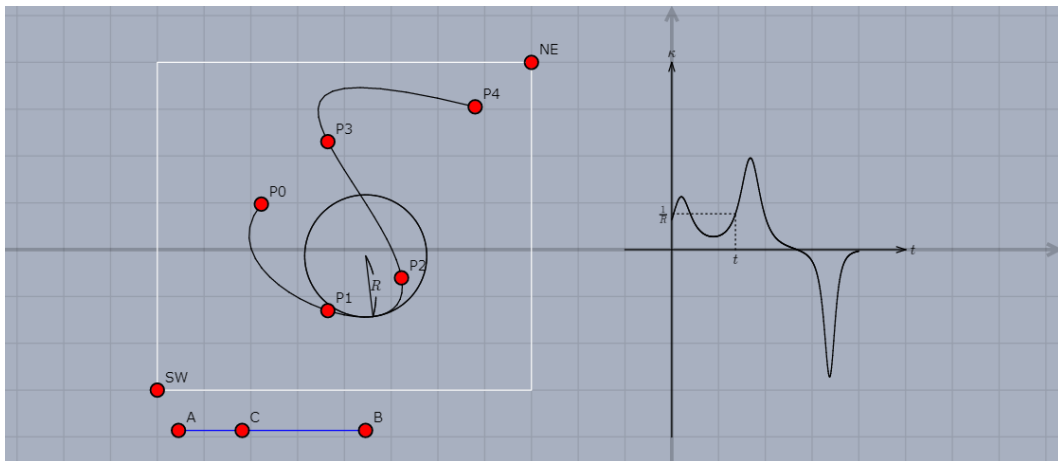


Figure 5: CindyJS content

3 Teaching materials for differential geometry course

In this section, we introduce teaching materials for differential geometry courses, made using $\text{K}_{\text{ET}}\text{Cindy}$. The aim of these materials is to help students intuitively understand the concepts of curvatures of curves and surfaces through figures generated using $\text{K}_{\text{ET}}\text{Cindy}$. For a detailed description of these concepts, see [12], for example.

3.1 Curvature of plane curves

Let $C(t) = (x(t), y(t))$ be a parametrized curve in \mathbb{R}^2 . Take a point P on this curve and consider Γ a unique circle which approximates the curve C in the best possible manner. More precisely, Γ is determined so that the derivatives coincide to those of the curve C up to the second order. This circle is called the *circle of curvature*, and its radius R is defined as the *radius of curvature* at P . In the case where the first and second derivatives vanish, we consider that the circle of curvature degenerates to a line and define the radius of curvature as infinity.

Now take the unit tangent vector e_1 to C at P in the direction where the parameter t increases and let e_2 be the unit vector at P obtained by rotating e_1 by $\frac{\pi}{2}$.

Then the *curvature* κ of the curve C at a point P is defined so that its absolute value is $\frac{1}{R}$ and its sign is positive when the circle of curvature lies on the left side of C , and negative otherwise. See Figure 6.

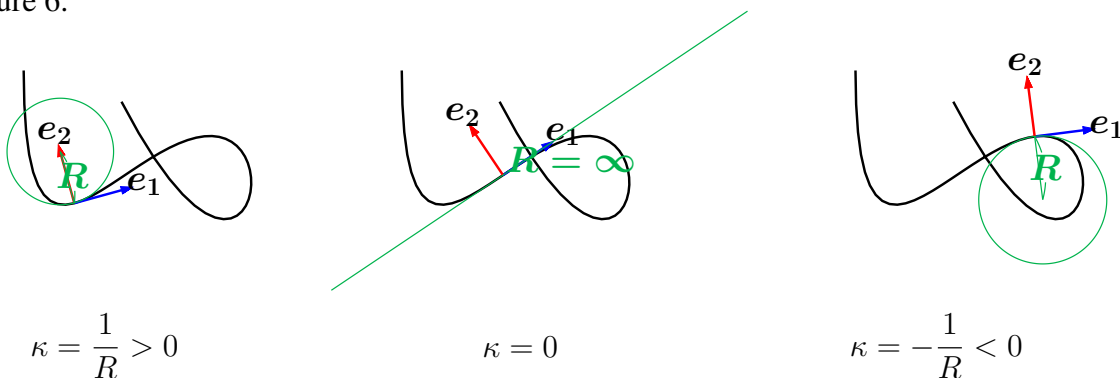


Figure 6: The sign of curvature

The following is an animation which illustrates the change of the circle of curvature and the curvature function of a curve:

Figure 7: Curve and its curvature

The components $x(t), y(t)$ of $C(t)$ are fourth order polynomials so that the curve $C(t)$ passes through given points P_1, \dots, P_5 . One can move these points interactively on the Cinderella screen and bend the curve to the desired shape. Figure 5 and [S1] are CindyJS output. Such figures can be given to students as auxiliary teaching material.

3.2 Curvatures of surfaces

Now consider a surface S in 3-dimensional Euclidean space. There are several concepts of curvatures associated with a surface S at an arbitrary point P on it:

1. *Normal curvature*
2. *Principal curvatures*
3. *Gaussian curvature and mean curvature*

Let us explain these concepts in order.

3.2.1 Normal curvature

Let S be a surface and P be a point on S . Take a unit vector e_1 tangent to S at P and the unit vector e_2 normal to S at P . Note that e_2 is uniquely determined for a fixed orientation of S . On the other hand, the choice of e_1 has freedom of 2π , and parametrized by $\theta \in [0, 2\pi]$ with respect to a fixed direction. Then there exists a plane Π_θ which passes through a point P and is spanned by e_1, e_2 .

The intersection with the surface S defines a plane curve on Π_θ , which is tangent to e_1 and normal to e_2 at P ; therefore, the curvature as a plane curve can be defined, which is called the *normal curvature* $\kappa_\nu(\theta)$ of S at P in the direction of e_1 . By definition, the normal curvature $\kappa_\nu(\theta)$ is a function on the circle S^1 . The following is a Flip type animation which consists of 48 frames (see [S2] for a complete version).

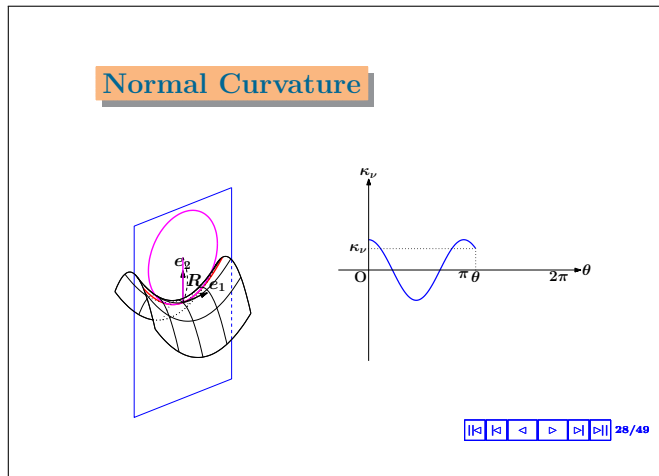


Figure 8: normal curvature

3.2.2 Principal curvatures

Since the circle is a compact space, there exist a maximum and a minimum of $\kappa_\nu(\theta)$. We call them *principal curvatures*, denoted by κ_1, κ_2 . The following slide explains the meaning of principal curvatures.

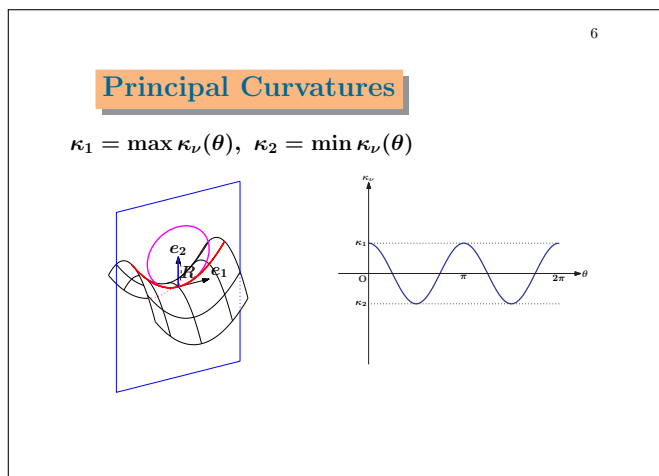


Figure 9: principal curvatures

3.2.3 Gaussian curvature

The product $\kappa_1\kappa_2$ of principal curvatures κ_1, κ_2 is called the *Gaussian curvature* of a surface S at a point P . For example, in the case of point P in the figure above, we can see that the Gaussian curvature K is negative as $\kappa_1 > 0$ and $\kappa_2 < 0$ hold.

On the other hand, at points where the surface is convex, as in the frame on the left in Figure 10, the Gaussian curvature at those points are positive as the signs of κ_1 and κ_2 coincide. In the case of a

cylindrical surface (Figure 10 right), we can easily see that $K = 0$ [S3].

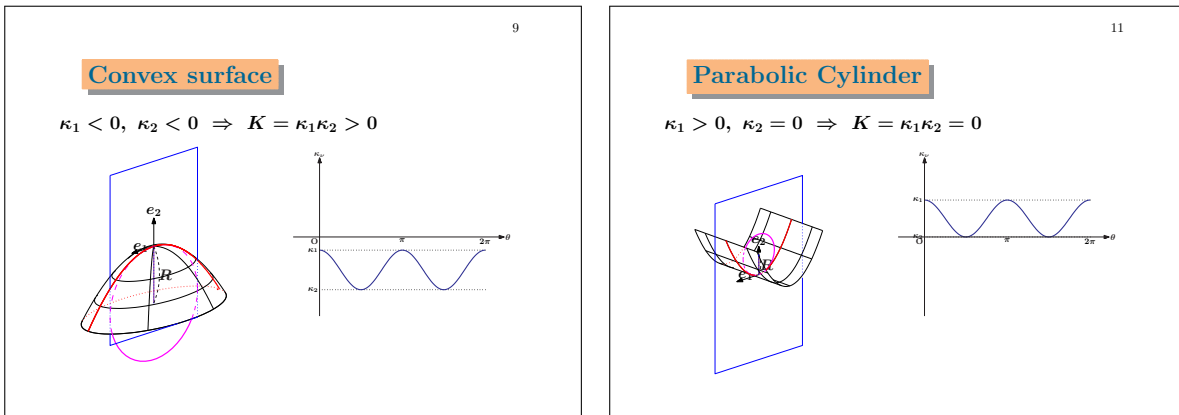


Figure 10

3.2.4 Mean curvature and minimal surfaces

If κ_1 and κ_2 are the principal curvature of a surface S at a point Pp in S , then the *mean curvature* H is defined as their mean $\frac{1}{2}(\kappa_1 + \kappa_2)$. A surface S is called *minimal* if its mean curvature vanishes at every point of S . It is easy to see that minimal surfaces have non-positive Gaussian curvature. A minimal surface can also be characterized as the surface with the least area among those with the same boundary. The following Flip type animation can explain this characterization of minimal surfaces (see [S4]).

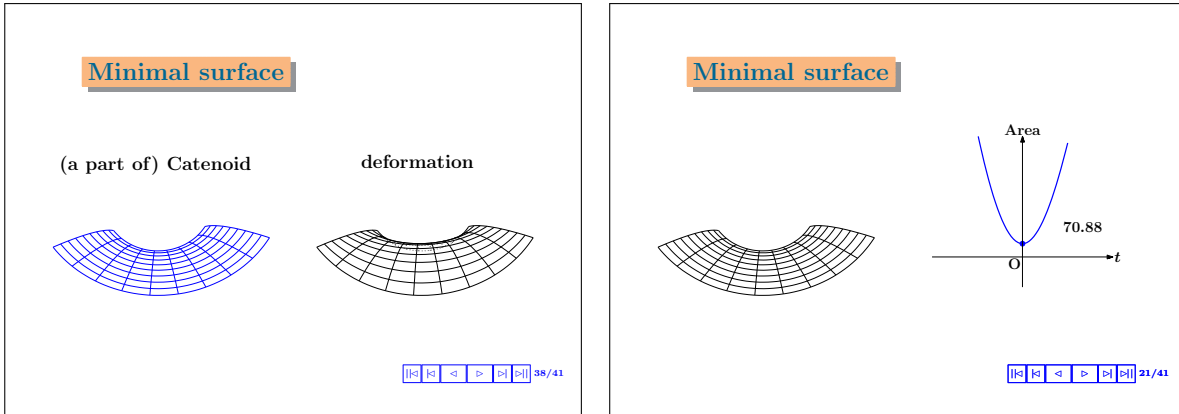


Figure 11: Minimal surface

3.2.5 Isometric deformation

Typical examples of minimal surfaces are the *helicoid* and the *catenoid*. Both of these surfaces have Gaussian curvatures of -1 and mean curvatures of zero, but they are embedded in space differently. A helicoid can be deformed continuously and isometrically into a catenoid, as the following animation illustrates.

4 Use of the materials in a class

In this section, we report on how the teaching materials in the last section were used in class. The first author has an elementary differential geometry course for third-year students majoring in information science or physics and who have studied calculus and linear algebra. Here we pick up a class at about the middle of the semester in which we used the materials described in this paper. The students had already learned about the curvatures of plane curves and parametric representation and the fundamental quantities of surfaces. The goal of the class was to help students understand the definitions and meanings of Gaussian and mean curvatures of surfaces and to get them to deduce the formula needed to calculate them.

The schedule of the 90-minutes class was as follows:

Time	Content	Device
5min	Review of geometric meaning of the curvature of a plane curve using materials we introduced in §3.1	projector
10min	Explanation of geometric meanings of curvatures using materials in §3.2	projector
30min	Mathematical formulation of the Gaussian and mean curvatures and derivation of the calculation formula for them	blackboard
10min	Revisit to geometric meanings of curvatures, with additional examples	projector
15min	Exercises on calculations of curvatures of surfaces explicitly defined by parameters	handout

Actually, the class was conducted roughly as planned. The following are the advantages that the authors found in using these teaching materials created with $\mathbb{K}\mathbb{E}\mathbb{T}\mathbb{C}\mathbb{i}\mathbb{n}\mathbb{d}\mathbb{y}$

- A variety of examples can be made by changing the parameters, function or viewpoint of one figure. In the class we reported on above, five different surfaces were considered to characterize the properties of Gaussian curvature.
- By using slides with figures and animations, we could show the outline of the theory in a short amount of time, which also helped the students to understand the mathematical arguments explained using a blackboard.
- As the part of the intuitive and geometrical illustration using PDF slides and that of mathematical argument using a blackboard were explained separately, both the students and the professor could avoid confusion and concentrate on each topic.
- By using prepared slide materials, time for explanation was reduced. As a result, the professor could introduce additional topics.
- All the PDF slides were uploaded to a web site, so that the students could download them. In fact, many students accessed the web site and used the slides for review.

5 Concluding remarks and future work

We have seen that by using KeTCindy we can produce visual materials which help students understand some aspects of the differential geometry of curves and surfaces. These materials include animation that will help learners to establish geometrical intuition of the concepts of curvatures. As KeTCindy generates TeX files, these figures can be presented both in slides and printed materials. If we use CindyJS in addition to these materials, the learning effect will be enhanced, and differential geometry will become much more familiar to students.

Acknowledgements

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Supplemental Electronic Material

Editor's Note: The [S1] link should be opened in Google Chrome and may require you to allow third-party JavaScript to show correctly.

- [S1] [Example of KeTCindyJS content](#)
- [S2] [PDF slides \(Flip type animation\): normal curvature](#)
- [S3] [PDF slides \(Flip type animation\): normal curvature of a convex surface and a parabolic cylinder](#)
- [S4] [PDF slides \(Flip type animation\): minimal surface](#)